Exact Zero-Class Geometry and Boundary Growth in Multiplication Tables Modulo N

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Abstract. This paper studies multiplication tables modulo N through the convex hulls of their residue classes. It introduces the corresponding residue-wise area invariant and analyzes two exact geometric structures inside the table. The first is the zero class, where divisor rectangles encode the effect of zero divisors and lead to a complete convex-hull description, a divisor-chain boundary, and an exact area formula. The second is the first boundary, the outer frame of the table, whose residue-wise four-point geometry gives an explicit lower model for the total area. Together, these results place exact zero-divisor geometry and cubic-order total area growth inside the same convex-hull framework.

1 Introduction

The multiplication table modulo N can be read as a finite planar object. Each cell is both a product modulo N and a lattice point in the positive-coordinate square. Fixing one residue selects a congruence-defined point set inside that square.

For an integer $N \geq 2$ and a residue $a \in \{0, ..., N-1\}$, write

$$A_{N,a} := \{(x, y) \in \{1, ..., N-1\}^2 : xy \equiv a \,(\operatorname{mod} N)\}. \quad (1)$$

and attach to this point set its convex-hull area

$$S(N, a) := \operatorname{Area}\big(\operatorname{conv}(A_{N,a})\big). \quad (2)$$

The total area invariant is

$$S(N) := \sum_{a=0}^{N-1} S(N, a). \quad (3)$$

Thus the multiplication table is viewed through the whole family of convex regions generated by its residue classes.

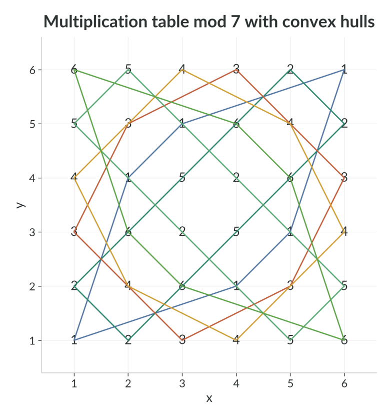

Figure 1: The multiplication table modulo 7 with convex hulls of all residue classes overlaid. Each colored polygon encloses the lattice points sharing one residue.

Every residue class has two elementary Euclidean symmetries. Commutativity gives the coordinate swap

$$(x, y) \mapsto (y, x),$$

and reflection through the center gives the half-turn

$$(x, y) \mapsto (N - x, N - y),$$

This is the rotation by 180° about $(N/2, N/2)$. Consequently, each convex hull conv $(A_{N,a})$ inherits both symmetries. The *orbit* of a point (x, y) means the four points (x, y), (y, x), $(N - x, N - y)$, $(N - y, N - x)$ obtained by these two symmetries.

For the unit class $a = 1$, the same two symmetry lines ($y = x$ and $x + y = N$) are recorded in Khan, Shparlinski, and Yankov [1], Proposition 2.1.

For fixed a, the sets $A_{N,a}$ are modular hyperbolas in the window $\{1, ..., N-1\}^2$; see Shparlinski's survey [2] for the surrounding literature. The convex hull of a modular hyperbola is itself the subject of Khan, Shparlinski, and Yankov [1], who studied the convex closure of the graph of modular inversions $xy \equiv 1 \,(\operatorname{mod} N)$, $a = 1$ in the present notation; convex-hull questions for these sets also appear in work of Konyagin and Shparlinski on vertex counts [3]. Related work studies the integer hull of all lattice points above a single hyperbola $xy \geq n$ [4], [5]. The object here is different: $A_{N,a}$ is finite, lies in the fixed window $\{1, ..., N-1\}^2$, and keeps only the points satisfying $xy \equiv$

$a \pmod N$. Much of this literature studies distribution, point counts, exponential-sum estimates, or vertex counts for individual residue classes, often in the coprime case where $\gcd(N, a) = 1$. In contrast, this paper fixes one modulus N and studies the convex-hull areas of all residue classes $a = 0, ..., N - 1$ at once, including the zero class and zero divisors. A nearby comparison point is the classical multiplication-table problem of Erdős [6], and in generalized form Koukoulopoulos [7], where the object is again a whole multiplication table but the invariant is the number of distinct ordinary products.

The two symmetries above already force a basic arithmetic fact about the area $S(N, a)$.

Proposition 1.1 (Integrality of residue-class areas).
For every $N \geq 2$ and every residue a, $S(N, a)$ is a nonnegative integer.

Proof. If conv $(A_{N,a})$ is degenerate, then its area is 0. Otherwise it is a lattice polygon. For every boundary lattice point (x, y), the half-turn gives a distinct boundary lattice point $(N - x, N - y)$. The boundary count is therefore even, and Pick's theorem [8] gives an integer area. □

In coprime or prime-base settings, when $\gcd(N, a) = 1$, the zero class disappears. For composite moduli, the zero residue appears along rows and columns whose coordinates carry complementary factors of N. A point belongs to $A_{N,0}$ when the two coordinates together supply all prime-power factors of N, and this condition gives the zero class its own geometry.

A separate exact structure, the boundary of the table, comes from its outer frame. This frame, consisting of the rows and columns with coordinate 1 or $N - 1$, contains a four-point pattern for each residue $a = 1, ..., N - 1$. Its convex-hull area can be summed over residues, giving a cubic lower model for the total invariant $S(N)$.

2 Zero-class geometry

For $a = 0$, the congruence $xy \equiv 0 \pmod N$ is a divisibility condition: the factors of N must be split between x and y.

Theorem 2.1 (Zero-class degeneracy criterion).
For every integer $N \geq 2$,

$$S(N, 0) = 0 \Leftrightarrow (N \text{ is prime}) \text{ or } N = 4.$$

Proof. There are three cases: prime N, $N = 4$, and composites $N > 4$.
If $N = p$ is prime, then no $x, y \in \{1, ..., p - 1\}$ is divisible by p, so $A_{p,0} = \varnothing$. For $N = 4$ one has

$$A_{4,0} = \{(2, 2)\},$$

so the hull is again degenerate.
Now let N be composite and $N > 4$. If N is even, then

$$(2, N/2), \quad (N/2, 2), \quad (N - 2, N/2), \quad (N/2, N - 2)$$

are distinct, belong to $A_{N,0}$, and form a rhombus of positive area. If N is odd, choose a proper divisor d of N and write $e = N/d$. Then

$$(d, e), \quad (N - d, e), \quad (d, N - e), \quad (N - d, N - e)$$

belong to $A_{N,0}$ and form a rectangle of positive area. Hence $S(N, 0) > 0$ for every composite $N > 4$. □

The degeneracy criterion isolates the only cases where the zero class has no area. For every remaining composite modulus, each proper divisor d supplies a complementary divisor N/d, and the corresponding zero-class points fill the corners of a rectangle.

For each divisor d of a composite integer N with $1 < d < N$, set

$$R_d := [d, N - d] \times \left[\frac{N}{d}, N - \frac{N}{d}\right].$$

This rectangle has x-coordinates from d to $N-d$ and y-coordinates from N/d to $N-N/d$.

Theorem 2.2 (Exact divisor-rectangle description).
If N is composite, then

$$\mathrm{conv}\ (A_{N,0}) = \mathrm{conv}\left(\bigcup_{\substack{d \mid N \\ 1<d<N}} R_d \right).$$

Proof. This is a two-inclusion argument. Let $(x,y) \in A_{N,0}$ and set $d = \gcd(x,N)$. If $d=1$, then $N \mid xy$ implies $N \mid y$, which is impossible because $1 \le y \le N-1$. Thus $d>1$. Since also $1 \le x \le N-1$, one has $d<N$, so d is a proper divisor of N.
Write $x = dq$ and $N = dw$ with $\gcd(q,w)=1$. Since $N \mid xy$, one has $dw \mid dqy$, hence $w \mid qy$, and therefore $w \mid y$. Thus $N/d \mid y$, so

$$d \le x \le N-d, \quad \frac{N}{d} \le y \le N - \frac{N}{d},$$

and $(x,y) \in R_d$. This proves

$$A_{N,0} \subseteq \bigcup_{\substack{d \mid N \\ 1<d<N}} R_d.$$

Conversely, for each proper divisor d, the four corners

$$\left(d, \frac{N}{d}\right), \quad \left(N-d, \frac{N}{d}\right), \quad \left(d, N-\frac{N}{d}\right), \quad \left(N-d, N-\frac{N}{d}\right)$$

all belong to $A_{N,0}$, so $R_d \subseteq \mathrm{conv}\ (A_{N,0})$. Taking convex hulls gives the reverse inclusion. $\square$
Figure 2 shows the distinction between the divisor-rectangle union and its convex hull.

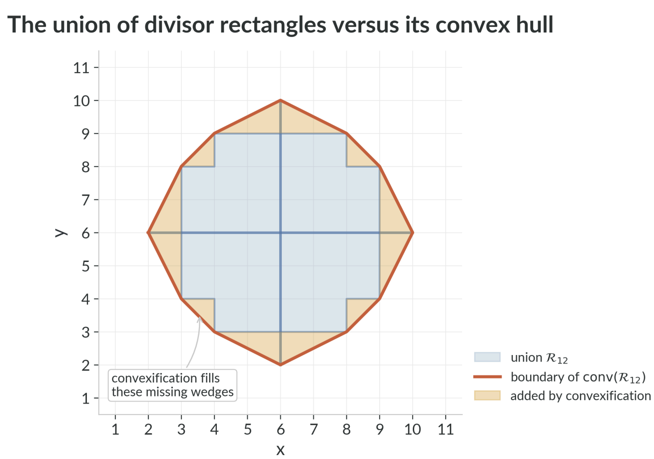

Figure 2: For $N = 12$, the union of divisor rectangles is not itself convex; convexifying that union gives the exact zero-class hull.

From the rectangle description, we can now read the lower boundary from the lowest available divisor corners.
Assume that N is composite. The factor-point set (or the divisor-point set) is

$$\mathcal{F}_N := \left\{ \left(d, \frac{N}{d}\right), \left(N-d, \frac{N}{d}\right) : 1 < d < N, d \mid N \right\}.$$

Let

$$p \coloneqq \min\{d > 1 : d \mid N\}.$$

For $p \le x \le N - p$, let $h_N(x)$ be the lower-boundary height of conv $(\mathcal{F}_N)$ at x.
List the proper divisors $d \le N/2$ in increasing order,

$$p = d_1 < d_2 < \cdots < d_\ell \le \frac{N}{2},$$

so that the lower-hull vertices are the divisor points $(d, N/d)$ taken in this order. For consecutive divisors $d < d'$, the chord between $(d, N/d)$ and $(d', N/d')$ has slope

$$\frac{\frac{N}{d'} - \frac{N}{d}}{d' - d} = -\frac{N}{dd'},$$

and these slopes increase strictly because the product dd' grows as the pair moves right ($d_1 d_2 < d_2 d_3 < \cdots < d_{\ell-1} d_\ell$). Thus the divisor points, taken in increasing order, form a convex chain — the left lower hull. If $\ell = 1$, there is only one divisor point on the left half. No consecutive-divisor chord is present, and the lower edge is the central horizontal segment from $(d_1, N/d_1)$ to $(N - d_1, N/d_1)$.
On the interval $[d, d']$, the lower hull is the chord through $(d, N/d)$ and $(d', N/d')$. With height variable $y = h_N(x)$, point $(d, N/d)$, and slope $-N/(dd')$, the point-slope equation is

$$h_N(x) - \frac{N}{d} = -\frac{N}{dd'}(x - d).$$

Solving for $h_N(x)$ gives the formula below. If N is odd, the reflected point $(N - d_\ell, N/d_\ell)$ gives a central horizontal segment; if N is even, then $d_\ell = N/2$ and no middle segment occurs. The right half follows from the left-right symmetry $h_N(N - x) = h_N(x)$.

Proposition 2.3 (Divisor-chain formula for the lower hull).
With the notation above,

$$h_N(x) = \begin{cases} N\left(\frac{1}{d} + \frac{1}{d'} - \frac{x}{dd'}\right), & d \le x \le d' \text{ for consecutive divisors } \ d < d', \\ \frac{N}{d_\ell}, & d_\ell \le x \le \frac{N}{2}, \quad N \ \text{ odd}, \\ h_N(N - x), & \frac{N}{2} \le x \le N - p. \end{cases}$$

Thus the lower boundary is a divisor chain, read in increasing order.
The same lower line determines the full zero-class hull. The upper boundary is obtained by the half-turn, so the filled hull is the region between the two polygonal chains.

Proposition 2.4 (Zero-class hull between two polygonal chains).
Assume that N is composite. Then

$$\text{conv } (A_{N,0}) = \{(x, y) : p \le x \le N - p, h_N(x) \le y \le N - h_N(x)\}.$$

Proof. By the divisor-rectangle theorem, it remains to identify the convex hull of $\bigcup_d R_d$. Every divisor rectangle lies on or above the lower chain because its lower corners belong to $\mathcal{F}_N$. Since the preceding derivation identifies a convex lower chain, convexifying the rectangle union does not create points below $y = h_N(x)$.
The lower line is reached because $\mathcal{F}_N$ is contained in the divisor-rectangle union. The half-turn supplies the upper boundary $y = N - h_N(x)$. The horizontal range is $p \le x \le N - p$, forced by the smallest proper divisor p and its reflected point $N - p$. $\square$
Figure 3 shows the lower chain, its reflected upper edge, and the vertical thickness used in the area formula.

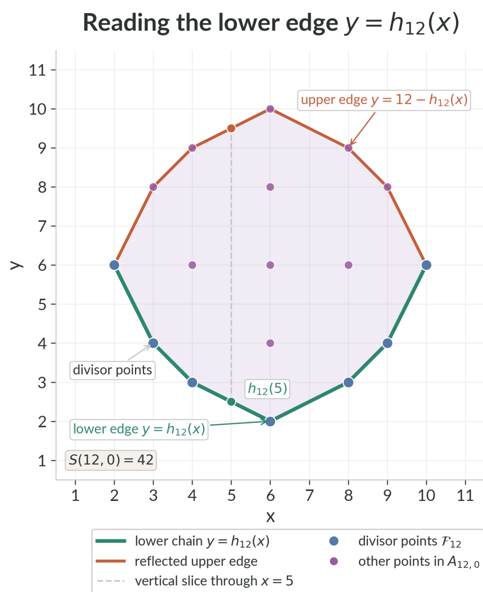

Figure 3: For $N = 12$, the divisor points determine the lower edge $y = h_{12}(x)$; the upper edge is obtained by the half-turn, and the zero-class hull is the region between them.

At horizontal coordinate x, the vertical thickness of this region is

$$(N - h_N(x)) - h_N(x) = N - 2h_N(x).$$

Integrating this thickness over $[p, N - p]$ gives the area. Equivalently, each reflected pair of intervals $[d, d']$ and $[N - d', N - d]$ contributes

$$2(d' - d)\left(N - \frac{N}{d} - \frac{N}{d'}\right).$$

If N is odd, the central horizontal segment contributes the additional strip

$$(N - 2d_\ell)\left(N - \frac{2N}{d_\ell}\right).$$

This gives the exact area formula.

Proposition 2.5 (Exact zero-class area).
With the notation above,

$$S(N, 0) = 2 \sum_{\text{consecutive} \;\; d<d'} (d' - d)\left(N - \frac{N}{d} - \frac{N}{d'}\right) + \varepsilon_N,$$

where

$$\varepsilon_N = \begin{cases} 0, & \text{if } \; N \; \text{ is even,} \\ (N - 2d_\ell)\left(N - \frac{2N}{d_\ell}\right), & \text{if } \; N \; \text{ is odd .} \end{cases}$$

3 First boundary and cubic growth

The multiplication table can also be read from the outside inward, one boundary layer at a time. The first layer is the outer frame: the bottom and top rows, and the left and right columns of the square. The next layer uses the rows and columns one step farther in, then the next layer moves inward again, until the square is exhausted.

Summing the hull areas of its first layer boundary orbits gives a lower model for the total area $S(N)$, and this model is already cubic in N.

For $N \geq 2$ and $a \in \{0, 1, ..., N-1\}$, define

$$B^{(1)}_{N,a} \coloneqq A_{N,a} \cap ((\{1, N-1\} \times \{1, ..., N-1\}) \cup (\{1, ..., N-1\} \times \{1, N-1\})),$$

and set

$$S^{(1)}(N, a) \coloneqq \text{Area } (\text{conv } (B^{(1)}_{N,a})), \quad S^{(1)}(N) \coloneqq \sum_{a=0}^{N-1} S^{(1)}(N, a).$$

By construction,

$$B^{(1)}_{N,a} \subseteq A_{N,a}, \quad S^{(1)}(N, a) \leq S(N, a), \quad S^{(1)}(N) \leq S(N).$$

For $1 \leq a \leq N-1$, the first-boundary points form the orbit of $(1, a)$:

$$(1, a), \quad (a, 1), \quad (N-1, N-a), \quad (N-a, N-1).$$

On each frame line the residue occurs once: $x = 1$ forces $y = a$; $x = N-1$ forces $(N-1)y \equiv a$, i.e. $y = N-a$; $y = 1$ forces $x = a$; and $y = N-1$ forces $x = N-a$. So $B^{(1)}_{N,a}$ is exactly this four-point orbit, with no other frame points. For $a = 1$ or $a = N-1$ (with $N \geq 3$), opposite points coincide and the hull degenerates to a segment. Otherwise the four points form a parallelogram centered at $(N/2, N/2)$. One side of this parallelogram from $(1, a)$ to $(a, 1)$ has vector $(a-1, 1-a)$ and length $\sqrt{2}(a-1)$; the adjacent side from $(a, 1)$ to $(N-1, N-a)$ has vector $(N-a-1, N-a-1)$ and length $\sqrt{2}(N-a-1)$. These two sides lie in perpendicular diagonal directions, so the parallelogram is a rectangle and the area is

$$2(a-1)(N-a-1).$$

The same formula gives 0 in the two degenerate endpoint cases. There are no zero residues on the first boundary, so $S^{(1)}(N, 0) = 0$.

Figure 4 shows the first-boundary model for $N = 7$: each residue $a = 1, ..., N-1$ appears as one orbit on the frame, with the nondegenerate orbits forming tilted rectangles.

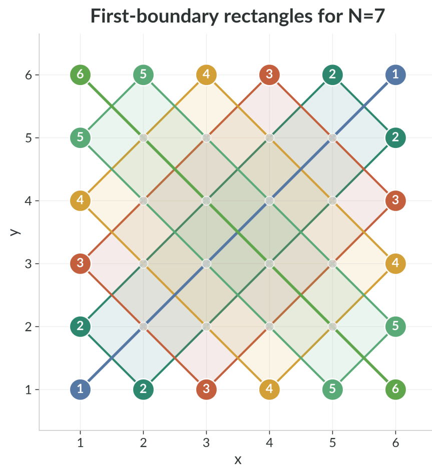

Figure 4: The first boundary for $N = 7$. The boundary residues are colored by residue, the interior points are grayed out, and each nondegenerate residue orbit determines a tilted rectangle. The endpoint residues 1 and 6 collapse to line segments.

Proposition 3.1 (Exact first-boundary formula).
For every $N \geq 2$,

$$S^{(1)}(N,0) = 0, \quad S^{(1)}(N,a) = 2(a-1)(N-a-1) \quad (1 \leq a \leq N-1).$$

Set $m = N-2$ and $b = a-1$. As a runs from 1 to $N-1$, the integer b runs from 0 to m, and the other side count is $m-b$. Hence

$$S^{(1)}(N) = 2\sum_{b=0}^{m} b(m-b).$$

Using the standard sums for $\sum b$ and $\sum b^2$ gives

$$2\sum_{b=0}^{m} b(m-b) = \frac{m(m+1)(m-1)}{3}.$$

Substituting $m = N-2$ gives the total first-boundary sum.

Theorem 3.2 (Exact total first-boundary sum).
For every $N \geq 2$,

$$S^{(1)}(N) = \frac{(N-3)(N-2)(N-1)}{3}.$$

Theorem 3.3 (Cubic-order theorem).
For every $N \geq 2$,

$$\frac{(N-3)(N-2)(N-1)}{3} \leq S(N) \leq N(N-2)^2.$$

In particular,

$$S(N) = \Theta(N^3).$$

Proof. By the exact first-boundary sum and the lower-model inequality $S^{(1)}(N) \leq S(N)$, the lower bound follows.
For the upper bound, every hull $\operatorname{conv}(A_{N,a})$ lies in $[1, N-1]^2$, whose area is $(N-2)^2$, so

$$S(N) \leq \sum_{a=0}^{N-1} (N-2)^2 = N(N-2)^2.$$

The two bounds imply $S(N) = \Theta(N^3)$. $\square$

4 Closing

The zero residue class has an exact divisor-controlled convex geometry: the hull is the convex hull of divisor rectangles, its lower boundary is the divisor chain, and its area is the corresponding divisor-chain sum.

The nonzero residue classes are treated in a subsequent preprint, where their hull vertices are constructed exactly by a divisor-lens method introduced there [9].

5 Author's note

The zero-class and zero-divisor geometry of this paper, the residue-class symmetries, and the first-boundary growth model are the author's. The proof that the zero-class hull is the convex hull of its divisor rectangles came out of a close collaboration with AI tools, whose contribution to the arguments and the exposition was substantial. Across the paper, AI tools also assisted with wording, phrasing, grammar, and error-checking during revision. The author guided and verified that work and is responsible for the mathematics and the final text.

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